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On the Equivalence of Different Order Parameters for Ising Ferromagnets

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In a recent note Barber showed, for a spin-$\frac{1}{2}$ Ising system with ferromagnetic pair interactions, that some critical exponents of the "triplet order parameter" $\langle\sigma_i\sigma_j\sigma_k\rangle$ are the same as those of the magnetization $\langle\sigma_i\rangle$. Here we prove such results for all odd correlations and dispense with the requirement of pair interactions. We also prove that the critical temperature T_c, defined as the temperature below which there is a spontaneous magnetization, is for fixed "even" spin interactions $\mathbf{J}_e$ independent of the way in which the "odd" interactions $\mathbf{J}_o$ approach zero from "above." This is achieved by using only the "simplest," Griffiths–Kelley–Sherman (GKS), inequalities, which apply to the most general many-spin, ferromagnetic interactions.

KEY WORDS: Ising ferromagnets; different order parameters; critical exponents; inequalities.

We consider an Ising spin system with general (many-spin) ferromagnetic interactions; for a bounded region Λ of a d-dimensional lattice the interaction energy of the spins in Λ is[(2)]

$$\begin{aligned} H(\boldsymbol{\sigma}_\Lambda) &= -\sum_{A\subset\Lambda} J_A\sigma_A \\ &= -\left(\sum h_i\sigma_i + \sum J_{ij}\sigma_i\sigma_j + \sum J_{ijk}\sigma_i\sigma_j\sigma_k + \cdots\right), \qquad J_A > 0 \qquad (1) \end{aligned}$$

Here $\boldsymbol{\sigma}_\Lambda = \{\sigma_i\}$, $i\in\Lambda$, $\sigma_i = \pm 1$, A is any subset of Λ, and $\sigma_A \equiv \prod_{i\in A}\sigma_i$. On the right side of (1) we have written out $H(\boldsymbol{\sigma}_\Lambda)$ more explicitly according to

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the number of sites $|A|$ in the set A and we have used the standard notation h_i, the magnetic field at the site i, for the one-body interaction J_A, $|A| = 1$.

When $J_A \geqslant 0$ for all A—"positive" ferromagnetic case—the expectation values of the spins at temperature $T = \beta^{-1}$,

$$\begin{aligned}\langle\sigma_A\rangle &= Z^{-1}\,\mathrm{Tr}\{\sigma_A \exp[-\beta H(\boldsymbol{\sigma}_\Lambda)]\} \\ &= \partial(\ln Z)/\partial(\beta J_A); \qquad Z = \mathrm{Tr}\{\exp[-\beta H(\boldsymbol{\sigma}_\Lambda)]\}\end{aligned} \tag{2}$$

satisfy the Griffiths–Kelley–Sherman (GKS) inequalities[2,3]

$$\langle\sigma_A\sigma_B\rangle - \langle\sigma_A\rangle\langle\sigma_B\rangle = \partial\langle\sigma_A\rangle/\partial(\beta J_B) \geqslant 0, \qquad J_C \geqslant 0 \quad \text{all } C \tag{3}$$

It follows from (3) that if there exist some strictly positive interactions $J_B > 0$ which "connect" all sites i in some set E such that these interactions *alone* would make the average of σ_E strictly positive at finite temperatures, then

$$\langle\sigma_E\rangle \geqslant K_E > 0 \tag{4}$$

where K_E can be taken independent of T for all temperatures $T \leqslant T_0 \leqslant \infty$. We shall assume from now on that this is true for all sets E, $|E|$ even. As an example, let E consist of the sites k and l and assume that $J_{ij} = J > 0$ for nearest-neighbor sites i and j. Then

$$\langle\sigma_k\sigma_l\rangle \geqslant [\tanh(\beta J)]^{|k-l|} \tag{5}$$

where $|k - l|$ is the length of the shortest "path" connecting k and l. The right side of (5) is simply the expectation value of $\langle\sigma_k\sigma_l\rangle$ when the only interactions are the nearest-neighbor ones along the path from k to l.

Suppose now that we adjoin to the set E another site i. We then have from (3) that

$$\langle\sigma_i\sigma_E\rangle \geqslant \langle\sigma_i\rangle\langle\sigma_E\rangle \tag{6}$$

$$\langle\sigma_i\rangle = \langle\sigma_i\sigma_E\sigma_E\rangle \geqslant \langle\sigma_i\sigma_E\rangle\langle\sigma_E\rangle \tag{7}$$

where we have used the fact that $\sigma_E\sigma_E = 1$. Using (4), we then have

$$K_E\langle\sigma_i\rangle \leqslant \langle\sigma_i\sigma_E\rangle \leqslant K_E^{-1}\langle\sigma_i\rangle \tag{8}$$

which is our basic inequality.

We shall now use (8) to obtain information about critical exponents of "odd"-order parameters, e.g., the triplet $\langle\sigma_1\sigma_2\sigma_3\rangle$ considered by Wood and Griffiths,[4] Baxter,[5] and Barber.[1] To do this we assume that the interactions are translation invariant, $J_A = J_{A+x}$, where $A + x$ is the set A translated by a lattice vector x, i.e., $h_i = h$, $J_{ij} = J^{(2)}(i - j)$, etc. Equation (1) now assumes the form

$$H(\boldsymbol{\sigma}_\Lambda, \mathbf{J}, b_\alpha) = \sum_{B \supset \{0\}} \sum_x J_B \sigma_{B+x}, \qquad J_B \geqslant 0 \tag{1$'$}$$

where $\{0\}$ designates the origin and the sum over x goes over all x such that $\{B + x\} \cap \Lambda$ is not empty, i.e., at least some of the sites in the set $B + x$ are in Λ. When not all sites of $B + x$ are in Λ, $\sigma_{B+x} = \prod \sigma_i \prod \sigma_j$, $i \in \{B + x\} \cap \Lambda$, $j \in \{B + x\} \cap \Lambda^c$, then we *either* set $\sigma_j = 0$ or $\sigma_j = 1$ for all $j \notin \Lambda$. These correspond, respectively, to zero and + boundary conditions, $b_\alpha = b_0$ or $b_\alpha = b_+$.[2,6] We also assume that there is a "cutoff" in the range of the interactions[2] so that we can let the region Λ increase to the whole lattice, i.e., take the thermodynamic limit, in a simple way.

It follows from the GKS inequalities that for $\mathbf{J} > 0$ the infinite-volume limit, $\Lambda \to \infty$, of the correlation functions $\langle\sigma_A\rangle(\beta, \mathbf{J}; b_\alpha, \Lambda)$ exist and are translation invariant.[2,6] We shall denote these limits by $\langle\sigma_A\rangle(\beta, \mathbf{J}; b_\alpha)$, $\alpha = 0$ or $+$, corresponding to b_0 or b_+ boundary conditions (b.c.). They are, by (8), monotone functions of $\mathbf{J}$, for $\mathbf{J} > 0$. In particular, $\langle\sigma_i\rangle(\beta, \mathbf{J}; b_\alpha) = m(\beta, \mathbf{J}; b_\alpha)$ is the magnetization per site. Letting $\mathbf{J} = (\mathbf{J}_o, \mathbf{J}_e)$ for "odd" and "even" interactions, then

$$\langle\sigma_A\rangle(\beta, \mathbf{J}_o, \mathbf{J}_e; b_\alpha) = (-1)^{|A|}\langle\sigma_A\rangle(\beta, -\mathbf{J}_o, \mathbf{J}_e; b_\alpha') \tag{9}$$

where $b_0' = b_0$, $b_+' = b_-$, "minus" b.c.

We now consider the behavior of the odd correlation functions in the limit in which all the odd interactions $\mathbf{J}_o \to 0$ while $\mathbf{J}_e$ remains fixed, e.g., in the simplest case in which $\mathbf{J}_o = h$ we wish to know the behavior of $\langle\sigma_A\rangle(\beta, h; b_\alpha)$ as $h \to 0+$. To answer this question we state the following lemmas, whose proofs follow from arguments similar to those used in the proofs of Lemmas 1 and 3 in Ref. 6.

Lemma 1. Let $\mathbf{J}' \geqslant 0$; then

$$\lim_{\mathbf{J}' \nearrow \mathbf{J}} \langle\sigma_A\rangle(\beta, \mathbf{J}'; b_0) = \langle\sigma_A\rangle(\beta, \mathbf{J}; b_0) \tag{10}$$

and

$$\lim_{\mathbf{J}' \searrow \mathbf{J}} \langle\sigma_A\rangle(\beta, \mathbf{J}; b_+) = \langle\sigma_A\rangle(\beta, \mathbf{J}; b_+) \tag{11}$$

Lemma 2. Let $\mathbf{J}' = (J_B', \hat{\mathbf{J}})$, $J_B' > 0$, $\hat{\mathbf{J}} \geqslant 0$; then

$$\begin{aligned}\langle\sigma_B\rangle(\beta, J_B, \hat{\mathbf{J}}; b_0) &= \lim_{\mathbf{J}_B' \nearrow \mathbf{J}_B} (\partial\psi/\partial J_B')(\beta, J_B', \hat{\mathbf{J}}) \\ &= \lim_{\mathbf{J}_B' \nearrow \mathbf{J}_B} \langle\sigma_B\rangle(\beta, J_B', \hat{\mathbf{J}}; b_+), \qquad J_B > 0\end{aligned} \tag{12}$$

and

$$\begin{aligned}\langle\sigma_B\rangle(\beta, J_B, \mathbf{J}; b_+) &= \lim_{\mathbf{J}_B' \searrow \mathbf{J}_B} (\partial\psi/\partial J_B')(\beta, J_B', \hat{\mathbf{J}}) \\ &= \lim_{\mathbf{J}_B' \searrow \mathbf{J}_B} \langle\sigma_B\rangle(\beta, J_B', \hat{\mathbf{J}}; b_0), \qquad J_B \geqslant 0\end{aligned} \tag{13}$$

where $\psi(\beta, \mathbf{J}) = \beta^{-1} \lim_{\Lambda\to\infty} \ln Z/|\Lambda|$ is the free energy per site in the thermodynamic limit, which is independent of boundary conditions.[2]

By convexity the right and left derivatives of ψ exist and coincide for almost all values of J_B (the exceptions being denumerable). At these values of J_B we also have that

$$\begin{aligned}\partial\psi(\beta, J_B, \hat{\mathbf{J}})/\partial J_B &= \lim_{\Lambda\to\infty} |\Lambda|^{-1} \sum_{\mathbf{x}} \langle\sigma_{B+x}\rangle(\beta, J_B, \hat{\mathbf{J}}; b_\Lambda, \Lambda) \\ &\equiv \langle\bar{\sigma}_B\rangle(\beta, J_B, \hat{\mathbf{J}}; b) \end{aligned} \tag{14}$$

where the sum over x is as in (1′) with $\{b_\Lambda\} = b$ a general set of boundary conditions[6,8] (corresponding to an arbitrary specification of σ_j for $j \notin \Lambda$), not just b_0 or b_+. At values of J_B where $\partial\psi(\beta, J_B, \hat{\mathbf{J}})/\partial J_B$ does not exist, we still have that $\langle\bar{\sigma}_B\rangle$ must lie between the left and right derivatives. Hence

$$\begin{aligned}\langle\sigma_B\rangle(\beta, J_B, \hat{\mathbf{J}}; b_0) &\leqslant \langle\bar{\sigma}_B\rangle(\beta, J_B, \hat{\mathbf{J}}) \\ &\leqslant \langle\sigma_B\rangle(\beta, J_B, \hat{\mathbf{J}}; b_+) \qquad \text{for} \quad J_B > 0\end{aligned} \tag{15}$$

and the strict $J_B > 0$ is needed only for the left-side inequality.

Let us now consider first the case where $\mathbf{J} = (h, \mathbf{J}_e)$, i.e., the only odd term in the energy is the one-body interaction, corresponding to an external field h. We can then define, for given fixed $\mathbf{J}_e$, a spontaneous magnetization critical temperature β_c^{-1} by

$$\lim_{h\to0+} m(\beta, h; b_\alpha) = \lim_{h\to0+} \frac{\partial\psi(\beta, h)}{\partial h} = m^*(\beta) = \begin{cases} 0, & \beta < \beta_c \\ >0, & \beta > \beta_c \end{cases} \tag{16}$$

where $m^*(\beta)$ is, by Lemma 1, equal to $\langle\sigma_i\rangle(\beta, \mathbf{J}_e; b_+)$. It is known[2,7,8] that β_c $[=\beta_c(\mathbf{J}_e)]$ must be greater than some minimum value $\beta_0 > 0$ and that for two and higher dimensions with positive (nonzero) nearest-neighbor pair interaction (this is sufficient but certainly not necessary) $\beta_c < \infty$.

It can be proven in some cases (see review in Ref. 8) and is believed to be "generally" true that $m^*(\beta_c) = 0$, and that the behavior of $m^*(\beta)$ near β_c, $\beta \geqslant \beta_c$, is characterized by a critical index b (usually called β), $m^*(\beta) \sim (\beta - \beta_c)^b$. It follows now from (8) that the same behavior is followed by *all odd* correlation functions.

Lemma 3. Let $\mathbf{J} = (h, \mathbf{J}_e) \geqslant 0$ be as in Eq. (1′) and let (4) hold. Define $\langle\sigma_A\rangle^*(\beta; b_\alpha) = \lim_{h\to0+}\langle\sigma_A\rangle(\beta, h; b_\alpha)$ with $\alpha = 0, +$; then, for all odd $|A|$,

$$K_1 m^*(\beta) \leqslant \langle\sigma_A\rangle^*(\beta; b_\alpha) \leqslant K_2 m^*(\beta), \qquad 0 < K_1, K_2 < \infty \tag{17}$$

In particular,

$$\langle\sigma_A\rangle^*(\beta; b_\alpha) = 0 \qquad \text{for} \quad \beta < \beta_c \tag{18}$$

and

$$\langle\sigma_A\rangle^*(\beta; b_\alpha) \sim (\beta - \beta_c)^b \qquad \text{for} \quad \beta \geqslant \beta_c \tag{19}$$

where b is the critical exponent for $m^*(\beta)$.

Remark. It can be readily shown, using Ginibre's proof of the GKS inequalities,[9] that $\langle\sigma_A\rangle(\beta, \mathbf{J}'; b_\Lambda, \Lambda) \leqslant \langle\sigma_A\rangle(\beta, \mathbf{J}; b_+, \Lambda)$ whenever $J_A \geqslant |J_A'|$. Hence for $\mathbf{J} = \{h, \mathbf{J}_e\}$, $\langle\sigma_A\rangle(\beta, h; b_\Lambda, \Lambda) \leqslant \langle\sigma_A\rangle(\beta, |h|; b_+, \Lambda)$ for all b_Λ, and thus we must have that

$$\lim_{h\to 0+}\sup \lim_{\Lambda\to\infty}\sup \langle\sigma_A\rangle(\beta, h; b_\Lambda, \Lambda) = 0 \qquad \text{for} \quad \beta < \beta_c, \quad |A| \text{ odd}$$

We can thus define β_c by the requirement that for $\beta < \beta_c$ all equilibrium states have vanishing odd correlations when $h = 0$.[10]

It should be noted here that for $\beta > \beta_c$ we do not know, for general $\mathbf{J}_e$, whether $\langle\sigma_A\rangle^*(\beta; b_0)$ equals $\langle\sigma_A\rangle^*(\beta; b_+) = \langle\sigma_A\rangle(\beta; b_+)$ for $|A| > 1$. It is only when $\mathbf{J}_e$ is restricted to pair interactions, or more generally to interactions satisfying the Fortuin–Kasteleyn–Ginibre[11] (FKG) inequalities, that one knows[6] that $\langle\sigma_A\rangle(\beta, h; b_e) = \langle\sigma_A\rangle(\beta, h)$, independent of *all* b.c. (not just b_0 or b_+) whenever $\partial\psi(\beta, h)/\partial h$ exists. Since this is true almost everywhere and $\langle\sigma_A\rangle(\beta, h; b_0)$ is monotone decreasing in h, the limits $h \to 0+$ of $\langle\sigma_A\rangle(\beta, h; b_\alpha)$ must coincide.

When $\mathbf{J}_e$ contains only pair interactions then by an extension of the Lee–Yang theorem we actually know[8,12] that $\langle\sigma_A\rangle(\beta, h)$ is analytic in h for $\operatorname{Re} h \neq 0$. We may then define the zero-field susceptibility by $\lim_{h\to 0+} \partial\langle\sigma_A\rangle(\beta, h)/\partial h$. In the more general case let

$$\chi(A, \beta; b_\alpha) = \lim_{h\to 0+} h^{-1}[\langle\sigma_A\rangle(\beta, h; b_\alpha) - \langle\sigma_A\rangle^*(\beta; b_\alpha)] \tag{20}$$

It follows now from (8) and (18) that $\chi(A; \beta, b_\alpha)$ has the same critical behavior for *all* odd $|A|$ as $\beta \to \beta_c$ from below, e.g.,

$$\chi(A, \beta; b_\alpha) \sim (\beta_c - \beta)^{-\gamma}, \qquad \beta_c \lesssim \beta$$

with γ independent of A. Since $\chi(i, \beta\colon b_\alpha)$ is, by Lemma 2, the same for $\alpha = 0, +$, γ is also independent of b_α.

The choice of the one-body potential (magnetic field h) as the sole symmetry-breaking interaction is of course arbitrary. It is not even clear a priori whether one can define for general $\mathbf{J}_e$ a unique critical temperature for symmetry breaking. This result follows, however, from our previous lemmas and the remark following Lemma 3.

Theorem. Let $\mathbf{J}_e \geqslant 0$ be a (finite) set of fixed, translation-invariant, even Ising spin interactions such that (4) holds. Then:

(a) There exist a β_c and positive constants C_1 and C_2 such that, for all $|A|$ odd,

$$\langle\sigma_A\rangle(\beta; b_+) = \begin{cases} 0, & \beta < \beta_c \\ >0, & \beta > \beta_c \end{cases} \tag{21}$$

and (for all $\beta \geqslant \beta_0 > 0$)

$$C_1 m^*(\beta) \leqslant \langle\sigma_A\rangle(\beta; b_+) \leqslant C_2 m^*(\beta)$$

where

$$m^*(\beta) = \langle\sigma_i\rangle(\beta; b_+) \tag{22}$$

(b) Let $\mathbf{J}_o \geqslant 0$ be odd interactions; then for $\mathbf{J} = (\mathbf{J}_e, \mathbf{J}_o)$, $\mathbf{J}_e$ fixed, and $|A|$ odd,

$$\langle\sigma_A\rangle(\beta; b_+) = \lim_{\mathbf{J}_o \searrow 0} \langle\sigma_A\rangle(\beta, \mathbf{J}_o; b_+) = \lim_{\mathbf{J}_o \searrow 0} (\partial/\partial J_A)\psi(\beta, \mathbf{J}_o) \tag{23}$$

where the last equality holds when $J_A > 0$, 1.

(c) For $\beta < \beta_c$ all infinite-volume equilibrium states for the interactions $\mathbf{J}_e$ have vanishing odd correlations.

(d) If $J_B > 0$, $|B|$ odd, then

$$\limsup_{\mathbf{J}_o \searrow 0} J_B^{-1}\langle\sigma_A\rangle(\beta, \mathbf{J}_o; b_+) = \chi_B(A, \beta; b_+) \tag{24}$$

satisfies the inequalities

$$C_1\chi_B(i, \beta; b_+) \leqslant \chi_B(A, \beta; b_+) \leqslant C_2\chi_B(i, \beta; b_+) \tag{25}$$

where, however, the values of χ_B may depend on the "path" $\mathbf{J}_o \searrow 0$.

Remarks. (i) These results can be extended further if we are willing to make some assumptions about interchanges of limits $\Lambda \to \infty$ and $\mathbf{J}_o \searrow 0$, which are certainly justified for the case of pair interactions.[8] We may then write

$$\lim_{\mathbf{J}_o \searrow 0} \partial\langle\sigma_A\rangle(\beta, \mathbf{J}_o; b_\alpha)/\partial J_B = \sum_{\mathbf{x}} \langle\sigma_A\sigma_{B+\mathbf{x}}\rangle(\beta; b_\alpha) \qquad \text{for} \quad \beta < \beta_c \tag{26}$$

Writing now $A = E \cup \{l\}$ and $B = E' \cup \{j\}$, where E and E' are even sets, we have, by the same arguments as led to (8), that

$$K_E K_{E'}\langle\sigma_i\sigma_{j+\mathbf{x}}\rangle(\beta; b_\alpha) \leqslant \langle\sigma_A\sigma_{B+\mathbf{x}}\rangle(\beta; b_\alpha) \leqslant (K_E K_{E'})^{-1}\langle\sigma_i\sigma_{j+\mathbf{x}}\rangle(\beta; b_\alpha) \tag{27}$$

We thus find that the different susceptibilities $\chi_B(A, \beta; b_\alpha)$ all have the same divergences as $\beta \to \beta_c$ from below.

(ii) When the FKG inequalities apply, e.g., only single and pair interactions, with $J(i - j) \geqslant 0$, then[6,8] for $h > 0$,

$$\langle\sigma_A\rangle(\beta, h; b) - \langle\sigma_A\rangle(\beta, 0; b) \leqslant \sum_{i\in A} [\langle\sigma_i\rangle(\beta, h; b) - \langle\sigma_i\rangle(\beta, 0; b)] \tag{28}$$

which can then be used in place of (7). One advantage of (28) is that it generalizes immediately, as does (6), to the case of "general" (non-spin-$\frac{1}{2}$) spins[13,14] when the free measure is compact, e.g., σ_i uniformly distributed in the interval $[-1, 1]$. It also seems possible to extend some of our results to the unbounded spin case of interest in field theory and to some two- and three-component systems,[15] but we shall not do that here.

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